

Effect of Closed Paths in Complex networks on Six Degrees of Separation and Disorder

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Abstract

Milgram Condition proposed by Aoyama et al. [11] plays an important role on the analysis of "six degrees of separation". We have shown[16], [17] that the relations between Milgram condition and the generalized clustering coefficient, which was introduced as an index for measuring the number of closed paths by us[6]-[10], are absolutely different in scale free networks [12] and small world networks[13],[14]. This fact implies that the effect of closed paths on information propagation is different in both networks. In this article, we first investigate the difference and pursuit what is a crucial mathematical quantity for information propagation. As a result we find that a sort of "disorder" plays more important role for information propagation than partially closed paths included in a network. Next we inquired into it in more detail by introducing two types of intermediate networks. Then we find that the average of the local clustering coefficient and the generalized clustering coefficients $C_{(q)}$ have some different functions and important meanings, respectively. We also find that $C_{(q)}$ is close to the propagation of information on networks. Lastly, we show that realizability of six degrees of separation in networks can be understood in a unified way by disorder.

keywords: Milgram Condition, Scale free network, Small world network, Generalized clustering coefficient, Entropy of degree

1 Introduction

In 1967, Milgram has made a great impact on the world by advocating "Six Degrees of Separation"[1]. After that, his research group made some social experiments to establish the concept[2], [3]. Recently Watts and his research group adduced evidence in support of the concept more intensively by experiments using e-mail[4], [5].

We have studied the mathematical structures of complex networks [6]-[10] to understand "six degrees of separation" advocated by Milgram[1]. Especially, we have developed an original formulation [6]-[10] based on "string formulation" developed by Aoyama[11] in order to study the effect of closed paths included in a network on information propagation.

We have proceeded with our study for scale free networks and small world networks in the basis of Milgram condition proposed by Aoyama [11]. As a result, it was proved that the generalized clustering coefficient, which takes on the responsibility of closed cycles in a network, has the opposite effect on information propagation in a sense of Milgram's experiment (The term of "information propagation" is always used thoroughly in this sense in this article) in the both networks. That is, the existence of closed paths in a network impedes the propagation of information in small world networks, but promotes it in scale free networks.

In this article, we first investigate the difference in both the networks and pursuit what is a crucial mathematical quantity for information propagation. As a result, we find that a sort of "disorder" plays more important role for information propagation rather than the generalized clustering coefficient and the average of local clustering coefficients. In fact, we show that an index responsible for information propagation is the entropy of the degree distribution and/or the standard deviation of degree data. Thus the opposite characteristic described above in the both networks can be understood in a unified way.

Next we further inquired into the subject in more detail by introducing two types of intermediate networks. One is the networks that link a regular network and a scale free network and second is the networks that link a sort of small world networks and the scale free network, It would be

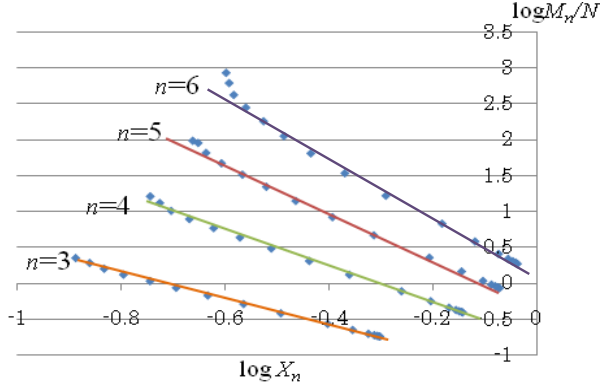


Figure 1: Sum of $C_{(p)}$ v.s. $\log M_q/N$ in small world networks for every separation number n .

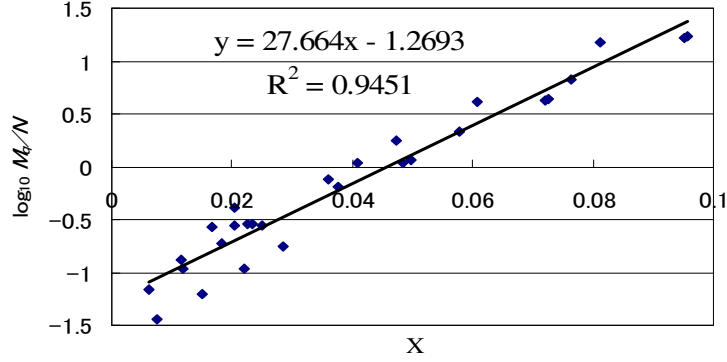


Figure 2: Separation number q v.s. $\log M_q/N$ in small-world networks with several α

expected that the transition between scale free network like property and small world network like one may be observed by such considerations.

Then we find that the average of the local clustering coefficients $C^{(3)}$ and the generalized clustering coefficients $C_{(q)}$, which are proposed by us[6]-[10] and have a generalization of the transitivity[22],[27], have some different functions and important meanings, respectively. There is a positive or a negative correlation between $C_{(q)}$ and M_6 in Milgram condition which will be defined in the next section, that is, $C_{(q)}$ and information propagation on the whole, but $C^{(3)}$ may grasp the transition between scale free \leftrightarrow small world (SF-SW transition). Then we find that small entropy of degree distribution and small standard deviation of degree data impede the propagation of information all in small world networks, scale free networks and the models introduced in this article. So the behaviors of M_6 or six degrees of separation in all networks discussed in this article are understood by disorder.

This article is planned as follows. After the introduction of the section 1, we describe the details of SF-SW transition in the section 2 and discuss disorder of networks by introducing the entropy of a degree distribution and the standard deviation of degree data. In section 3, we introduce two types of intermediate networks and estimate some statistical and global indices of the networks such as the average of local clustering coefficients, the generalized clustering coefficient, the average shortest path length and a quantity related to six degrees of separation by computer simulations. Moreover we estimate the degree of disorder of the two types of networks. In consequence, we show that the behaviors of M_6 or six degrees of separation in all networks discussed in this article are understood in a unified way by disorder.

2 Scale free networks vs. Small world networks

2.1 Milgram Condition in Scale free networks vs. Small world networks

According to Aoyama[11], we call j -string string like part consisting of j nodes in a network and S_j is the number of j -string in the network. \bar{S}_j is the number of non-degenerate j -strings [6]-[8], which are closed strings or open strings. By using it, We have defined the generalized clustering coefficient $C_{(q)}$ [6]-[8] as

$$C_{(q)} = \frac{2q \times \text{number of polygons}}{\text{number of connected } q\text{-plets}} = \frac{2q\Delta_q}{\bar{S}_q}, \quad (1)$$

where Δ_q is the number of polygonal structures with q edges in a network. This definition is the same as usual global clustering coefficient or "transitivity"[15] at $q = 3$ and gives its extensions for $q > 3$. This is different from the original clustering coefficient $C^{(3)}$ introduced by Watts[13],[14], which is the average of the local clustering coefficients;

$$\begin{aligned} C_i^{(3)} &= \frac{\text{number of pairs of neighbors of node } i \text{ that connected}}{\text{number of pairs of neighbors of node } i}, \\ C^{(3)} &= \frac{1}{N} \sum_i C_i^{(3)}, \end{aligned} \quad (2)$$

where N is the network size which is defined by the number of nodes in a network. A series of $C_{(q)}$ can be described by the adjacency matrix in a systematic way [6]-[8]. Then Milgram condition for n -th degrees of separation in a network with the size N is given by the following formula [11];

$$M_n \equiv \frac{\bar{S}_n}{N} \sim O(N). \quad (3)$$

When this condition is satisfied, it means that n -th degrees of separation becomes possible in the network. As the left hand side of eq. (3) becomes larger, it becomes easier to realize n -th degrees of separation.

In the fig.1 and fig.2, the relations between M_n/N and

$$X_n \equiv \sum_{p=3}^n C_{(p)} \quad (4)$$

are shown for small world networks constructed by Newman-Watts model [25],[26] and scale free networks constructed by the configuration like model where self-edges and multiedges are excluded in the original configuration model[27].

Fig.1 and fig.2 show that as the generalized clustering coefficient $C_{(q)}$ grows larger, it becomes easy for information to propagate in scale free networks, while it is difficult for information to propagate for larger $C_{(q)}$ in small world networks, provided, however, that data points with large M_6/N are those at small scaling exponent α and on the contrary the data points with small M_6/N are those at large α in fig.2. The same behaviors also apply for $C^{(3)}$.

The aspect of them is quantitatively shown as

$$M_n \sim B_n(X_n)^{-a}, \quad \text{for small world networks} \quad (5)$$

$$M_n \sim \exp(bX_n), \quad \text{for scale free networks} \quad (6)$$

where B_n , $a > 0$ and $b > 0$ are some constants[16], [17]. $C_{(q)}$ depends on the scaling exponent α in scale free networks and as α becomes smaller[16], $C_{(q)}$ becomes larger, which means that it becomes easier for information to propagate. This fact is also suggested in the configuration model where the average path length L for scale free networks with N nodes is given[23] by

$$\begin{aligned} L &\sim \log \log N, & \text{for } 2 < \alpha < 3, \\ &\sim \log N / \log \log N, & \text{for } \alpha = 3, \\ &\sim \log N, & \text{for } \alpha > 3, \end{aligned} \quad (7)$$

and the average of the local clustering coefficients $C^{(3)}$ is given [22] by

$$C^{(3)} = \frac{(\langle k^2 \rangle - \langle k \rangle^2)}{\langle k \rangle^3 N}, \quad (8)$$

where $\langle k \rangle$ is the average degree and so on. Eq.(7) means that the smaller α is, the smaller L is at a fixed N , so that information is considered to be easy to propagate over a network at smaller α . As for $C^{(3)}$, it is suggested to be large at small α from eq. (8), because $\langle k^2 \rangle$ notably becomes larger than $\langle k \rangle$ at small α by the cut-off theory[24] for which a brief review is given in the appendix. This consideration is consistent with the results of the approximate configuration model used in our analysis.

As said above, the effect of the generalized clustering coefficient, which bears a responsibility on how many closed paths there is in a network, on propagation of information is just opposite in the scale free networks and the small-world networks. How can we understand this some strange phenomenon at a glance? This will be discussed in the next subsection.

2.2 Disorder

We focus on when M_n is large in the both networks. M_n becomes larger at large rewiring ratio q in small world networks and then the networks come closer to random networks. Increasing the rewiring ratio makes information propagation urge in small world networks. As for scale free networks constructed by the configuration model, M_n becomes larger at small scaling exponent α and so decreasing α makes information propagation urge. Then α decreases, it also makes networks becomes disordered or homogeneous in degree. Thus we conjecture that the crucial point for information propagation would be a sort of "disorder". In both cases, "disorder", which is not yet defined appropriately, increase at least, as M_n becomes larger. We consider two indices for studying information propagation on networks, the standard deviation σ and the entropy as candidates of an amount of disorder.

The standard deviation σ of degree data is defined by

$$\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2. \quad (9)$$

The entropy is known as a concept that represents a sort of disorder for physics or indefiniteness for information theory. Some similar concepts of entropy regarding networks have been proposed[28],[30],[31]. The entropies introduced by [29] and [28] among them have analogous properties. In this article, we consider the entropy of degree distribution introduced by [29] as an index relevant to disorder of networks;

$$S = - \sum_{k=1}^{k_{max}} P(k) \log_2 P(k), \quad (10)$$

where $P(k)$ is the degree distribution.

For scale free networks, the entropy S is estimated as the following;

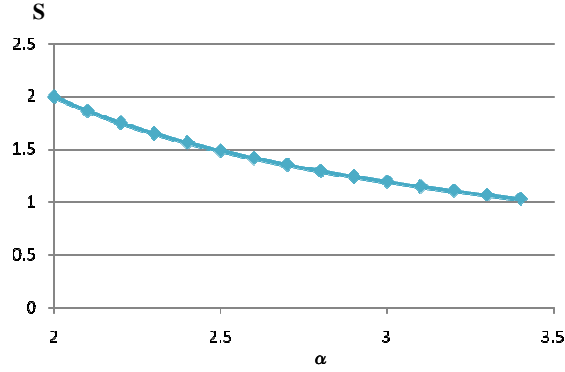
$$S = - \int_{k=k_{min}}^{\infty} dk P(k) \log_2 P(k) = \int_{k=k_{min}}^{\infty} dk c k^{-\alpha} \log_2 (c k^{-\alpha}) = - \left(\log_2 \left(\frac{\alpha-1}{k_{min}} \right) + \frac{\alpha}{1-\alpha} \right), \quad (11)$$

where $\alpha > 1$ and eq.(18) given in the appendix is used in the last equality. A typical data driven from eq.(15) in the appendix at $k_{min} = 4$ are plotted in fig.3. As is shown in this figure, when α increases, the entropy S decreases.

S has the minimum value in small world networks at $q = 0$, because all nodes have the same degree at $q = 0$, that is, the network becomes a regular circle lattice (WS lattice). It is also well known that an entropy has larger value for random data. The larger q is, the larger S is. σ also takes the minimum value 0 at $q = 0$ in small-world networks, because all nodes have the same degree at $q = 0$. $q \neq 0$ leads to large deviation in degree so that σ has non-zero value.

For scale free networks, we get from eq.(32)-(36) in the appendix,

$$\begin{aligned} \langle k \rangle^2 &\sim \left(\frac{\alpha-1}{2-\alpha} \right)^2 k_{min}^2 N^{\frac{4-2\alpha}{\alpha-1}}, & \text{for } 1 < \alpha < 2, \\ &\sim \left(\frac{\alpha-1}{2-\alpha} \right)^2 k_{min}^2, & \text{for } \alpha > 2, \end{aligned} \quad (12)$$



Entropy vs. scaling exponent in scale free networks for $k_{min} = 4$.

Network	α	q	M_n	C	S	σ
Small World		large	large	small	large	large
Scale Free	small		large	large	large	large

Table 1: Small World vs. Scale Free networks

and

$$\begin{aligned}
\langle k^2 \rangle &\sim \frac{\alpha-1}{3-\alpha} k_{min}^2 N^{\frac{3-\alpha}{\alpha-1}}, & \text{for } 1 < \alpha < 2, \\
&\sim \frac{\alpha-1}{3-\alpha} k_{min}^2, & \text{for } 2 < \alpha < 3.
\end{aligned} \tag{13}$$

We can get a numerical result for σ^2 based on eq. (9). Fig.4 shows that σ takes larger value at lower α except for near a singular point. The small peak at $\alpha \sim 3$ in fig.4 is due to the singularity inhered in $\langle k^2 \rangle$ shown in eq.(13). Such a singularity is idealized mathematical object and does not appear in real networks.

Evaluated characters up to now in both networks are summarized in Table 1 where C originally shows $C_{(3)}$ but $C^{(3)}$ also shows similar behavior. There are no quantities that correlate with C but σ and S behave like M_n qualitatively in both the networks. So it is noticed that it is σ and S that are relevant to information spreading, that is, Milgram condition. As the standard deviation and the entropy of degree distribution become larger, M_n becomes larger, too. Thus it is disorder relevant to information spreading in small world networks and scale free networks in common. As a consequence, we find a primary common factor in both networks.

3 Other networks inspired by SF and SW networks

3.1 Two intermediate networks

In order to investigate further the phenomenon that $C_{(q)}$ has the opposite effect on information propagation in both networks, we introduce the following two types of intermediate networks linking scale free networks and small world or regular networks. A small world network is an intermediate network between a random network and a regular network in some way, which are controlled by a rewiring parameter q . The random network can be a disordered network in the meaning that its entropy is large. On the contrary, the configuration models are interpreted as intermediate networks between a disordered network that has a large entropy and Barabashi-Albert model[12] in some way, which are controlled by a scaling exponent α . In this article, the term of "Barabashi-Albert model (BA model)", carries the connotation of scale free networks with scaling exponent $\alpha = 3$, even though they are not constructed by using the preferential attachment. Provided that scale free networks with α near zero can not studied practically because of bad singularity in $\langle k \rangle$ and so on.

So random networks are considered to be changed in two ways by parameter q and α , with the result that two types of changed networks (scale free networks and small world networks) have the

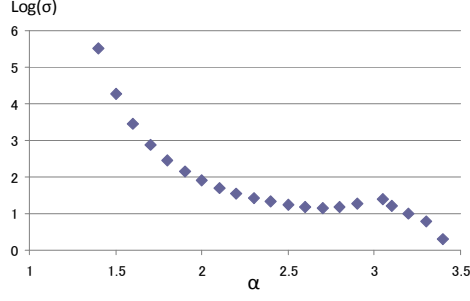


Figure 3: Standard deviation in scale free networks for $N = 400$.

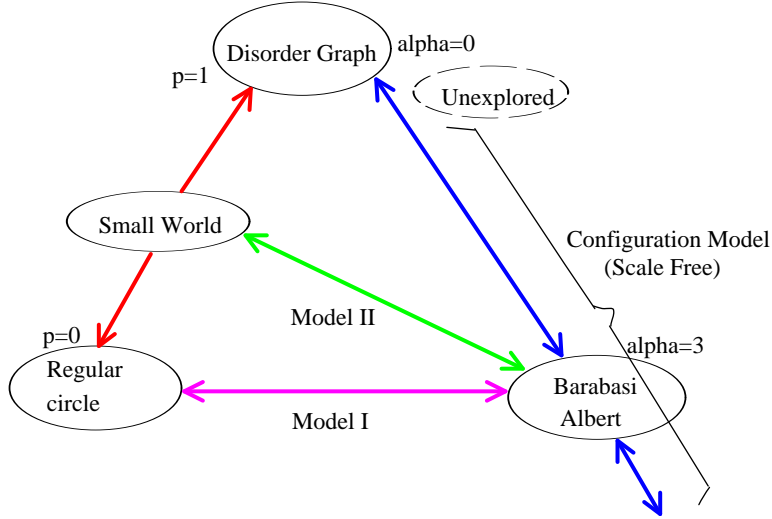


Figure 4: The relation among networks inspired by scale free and small world networks.

opposite behavior in the effect of closed paths on information spreading. The outline is described in fig. 5.

We are motivated to explore further two types of networks from the figure 5. They are called Model I and II, that are intermediate networks between small world networks and BA model and between regular networks and BA model, as described in fig.5, respectively. By investigating their intermediate networks, the critical phenomenon may be observed relevant to opposite behavior of $C(q)$ in small-world networks and scale free networks. The constructions of these two types of networks are as follows.

1. regular networks \rightarrow scale free networks (Mode I)
 Preferentially attach $N - N_0$ nodes with the degree k to nodes on a circle lattice (WS-lattice) with N_0 nodes which has all degree k_0 . $N - N_0$ is a control parameter in this model.
2. Scale free networks \rightarrow small world networks (Model II)
 Attach an edge between two nodes that connects to a hub, and instead deletes an edge attached with the hub but except for the edge between the hub and two nodes connected at present, to conserve the total number of edges in the considering network. This procedure are done with probability P for each node, which means that PN nodes follow this procedure in the order of degree rank, and makes the clustering coefficient large. The rewiring ratio P is introduced as a control parameter for clustering. There are a few attempts for constructing tunable scale free networks[29],[20],[21] that have small average path length and relatively large clustering coefficients. We do not accept them because they basically have a power low in the degree distribution that does not reflect the property of small world network.

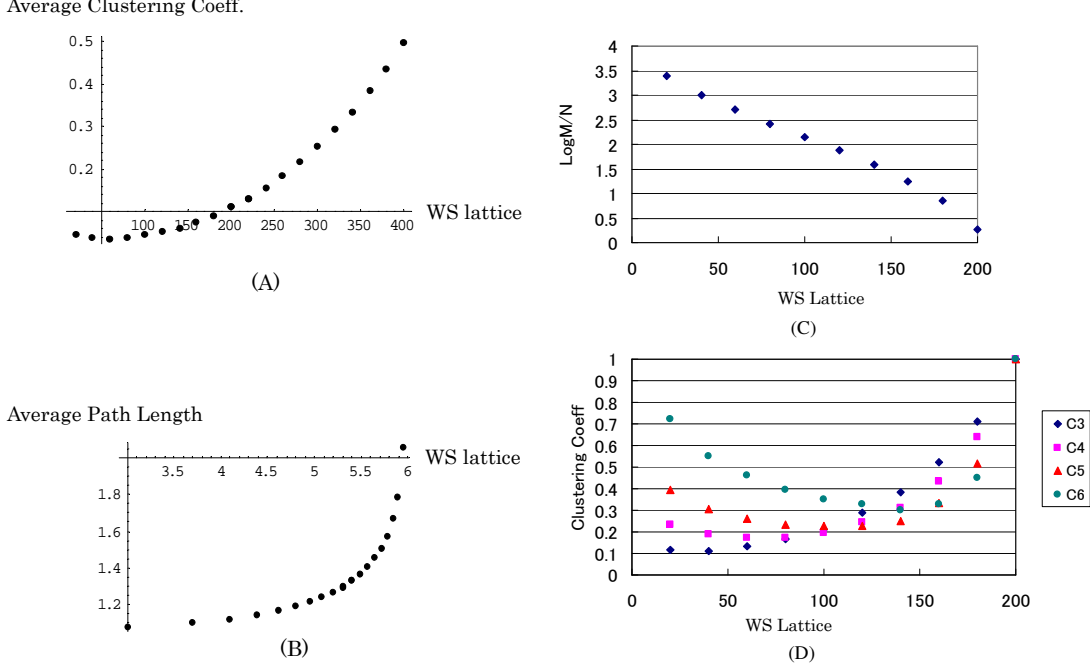


Figure 5: Average of local clustering coefficient (A), average path length (B), Milgram condition M_6/N and generalized clustering coefficient in the Model (I).

The procedure in the model II, however, really corresponds to the third step in the model introduced by B.Wang et al. as referred later on[29].

Since small world networks are a sort of the intermediate networks between a regular network and a random network, it will be of interest in intermediate networks between scale free networks and regular networks, instead of random networks. It gives a motivation for studying Model I. It will be also of interest to see what phenomena appear in the middle between scale free networks and small world networks with a large clustering coefficient, since $C_{(q)}$ has the opposite effect on information propagation in both networks. It is the reason to investigate Model II.

3.2 Simulation Results of Model I

(A), (B), (C) and (D) of fig. 6 show the average of local clustering coefficient $C^{(3)}$, L , sixth Milgram condition M_6/N in Milgram condition and the general clustering coefficients $C_{(q)}$ for $3 \leq n \leq 6$ vs. the initial size N_0 of WS lattice in the model I, respectively. The each plot is the average over ten rounds at $N = 400$ for (A) and (B) or over five rounds at $N = 200$ for (C) and (D) with $k = k_0 = 4$. The general clustering coefficients $C_{(q)}$ in fig.6 (D) are normalized as to be 1 at $N = 200$. As it has been shown that the behavior M_6/N is well correlated with that of the average path length L [16],[17], we also estimated L , which has an advantage over M_n/N from the perspective of computational complexity, besides M_n . The reason that M_6/N and $C_{(q)}$ are evaluated together over five rounds at $N = 200$ is that almost same computational complexity is needed in calculating them. We, however, consider that the stable results are given even in these cases.

As the proportion of preferential attachment in the network construction increases, that is, $N - N_0$ to N_0 , increases, the network gets closer to the scale free network. Though we find that L and $C^{(3)}$ increase together from (A) and (B) in fig.6, M_6/N in Milgram condition decreases in fig.6 (C), as N_0 grows larger. In other words, as $C^{(3)}$ increases, L becomes large but M_6/N decreases. Thus the increasing of small size closed cycles retard the spreading of information over networks. While the same behavior is also found in fig.6 (D), the larger size closed cycles ($C_{(q=5,6)}$) have not any definite influence on the spreading of information. $C_{(q)}$ with larger q temporarily decreases and then increases with N_0 . We infer that this behavior of $C_{(q)}$ makes the increase of L slow

in the small N_0 area in fig.6 (B). Anyway, information spreading over networks is closely bound up with small size cycles.

These properties are nearly the same as those of small world networks. In general small world networks as also shown in fig.1, the increase of closed cycles urges the increase (decrease) of L (M_n/N), and so closed cycles retards information propagation.

As a consequence, we can not, however, find any indications of transition phenomena between scale free networks and small world networks in this approach, where we bring scale free networks (BA model) into small world networks through regular networks (see fig.5). When transforming a network from BA model to a regular network (WS lattice), and further from small world small world to small world in fig.5, $C^{(3)}$ or $C_{(3)}$ become larger and M_6/N gets smaller monotonously between them.

3.3 Simulation Results of Model II

(A) , (B) , (C) and (D) of fig. 7 show $C^{(3)}$, L , M_6/N and $C_{(q)}$ for $3 \leq n \leq 6$ vs. the rewiring ratio P in the model II, respectively. The each plot is estimated in the average over five rounds at $N = 500$ for (A) and (B) and the average over five rounds at $N = 200$ for (C) and (D) by the similar reason to the previous subsection. The initial scale free network is made based on BA model with the scaling exponent $\alpha = 3$ [12]. On the whole, as the rewiring ratio increases, M_6/N and $C_{(q)}$ increase together in (C) and (D) of fig.7. So $C_{(q)}$, which conveys the information about how many cycles there are in a network, works to urge information propagation. This disposition looks like scale free networks. But $C^{(3)}$ becomes larger up to about $P = 0.4$ and becomes smaller for larger P than 0.4.

We analyze P dependent behaviors of the mathematical quantities represented in fig. 7 at full length.

1. $0 \leq P \leq 0.2$

M_6 , $C_{(q>3)}$ and L are almost constant but $C^{(3)}$ and $C_{(3)}$ grow larger together. It is reasonable that $C_{(3)}$ grows larger by the construction. So the effect of small size cycles in the networks does not reflect on information propagation.

2. $0.2 \leq P \leq 0.4$

At a little larger $P = 0.2$, M_6 and $C_{(q>3)}$ and L are leap little bit. They are, however, almost constant basically in this region of P . The increment of $C^{(3)}$ slows down. The value of L shows no great difference as usual.

3. $0.4 \leq P \leq 0.6$

In this region of P , M_6 make a gradual ascent and L make a gradual descent, as P becomes larger. Though $C_{(q \geq 3)}$ are almost constant, $C^{(3)}$ begins decreasing. The relation between $C^{(3)}$ and M_6 or L is like that of small world networks. $C_{(q \geq 3)}$, have not any effect on the behavior of M_6 or L .

4. $0.6 \leq P < 1.0$

M_6 is on the notable increase and L is on the notable decrease in this region. Though $C_{(q \geq 3)}$ is increasing, $C^{(3)}$ is slightly decreasing. While the behavior looks like small world networks from the perspective of $C^{(3)}$, the relation looks like scale free networks from the perspective of $C_{(q \geq 4)}$.

It is surmised that the reason that the increase of $C_{(3)}$ gradually slows down with P in fig.7 is that the triangles constructed so far begin to destroy when edges are taken away from a hub at large P . $C_{(3)}$ actually increases during small P in fig.7 (D).

Since increase of $C_{(q)}$ gives rise to the increase of M_6 on the whole, there are a positive correlation between $C_{(q)}$ and M_6 . In the meanwhile, since $C^{(3)}$ increases at $0 \leq P \leq 0.4$ with M_6 but contrariwise $C^{(3)}$ decreases at $0.4 \leq P$ with M_6 . However, fig.7 (A) shows a little strange behavior. Since BA scale free network is brought to "samall world networks" in such meaning proposed by Watts-Strogaz in Model II, which have a large clustering coefficient and a small average path length, this fact shows that the average local clustering coefficient $C^{(3)}$ may get some idea of SF-SW transition even at a fixed scaling exponent. But note that we can not completely affirm that it is so in the cases, where $C^{(3)}$ and $C_{(3)}$ do not show the same behavior. Thus it is considered that

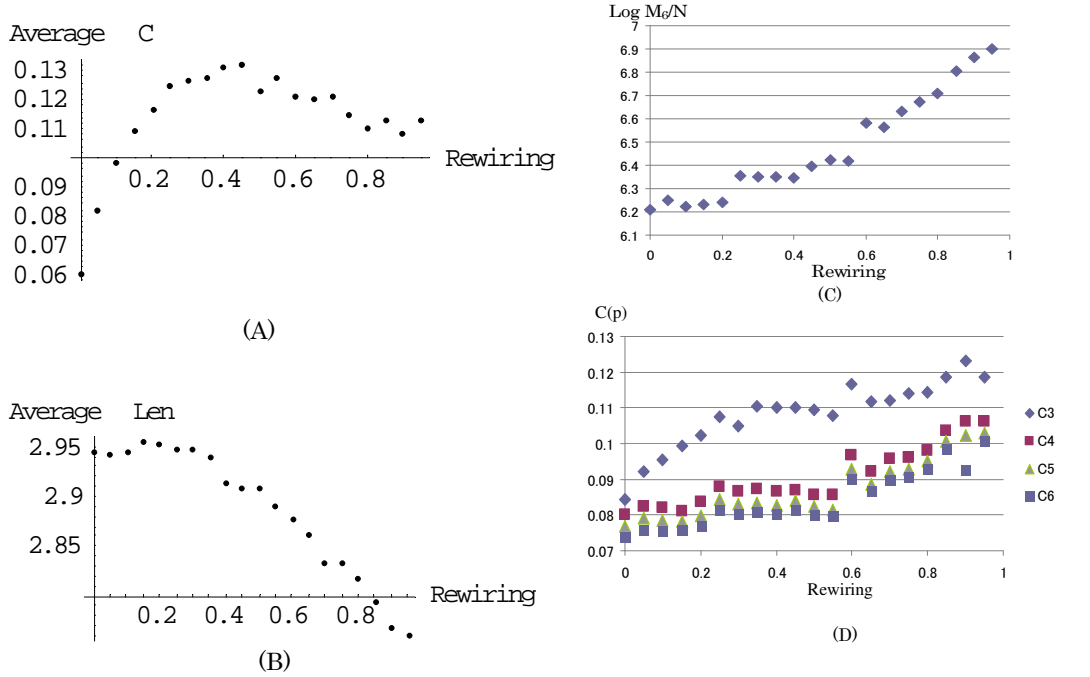


Figure 6: Average of local clustering coefficient(A), average path length (B), Milgram condition M_6/N and generalized clustering coefficient in Model (II).

$C^{(3)}$ and $C_{(q)}$ have different implications in this phenomena, respectively. While the small world network like behaviors in the relation between L and $C^{(3)}$ have been pointed out in the tunable scale free model proposed by Hølem and Kim[20], the result corresponds to the behavior in larger P area in our model.

In Model II, we can not also see an obvious transition between the scale free network like behaviors and the small world network like ones in the effect of closed paths on information propagation. Such characteristic of scale free networks observed in fig.2, that is, the relation between L and $C_{(q)}$, appeared through varying the scaling exponent of the networks [16],[17] (note that since the network construction is based on BA model in the Model II (and also I), the scaling exponent is the fixed value 3, even though the networks get near to a scale free network). So we speculate that a crucial point of explaining the fact that the closed paths (larger $C_{(q)}$) expedited the propagation of information in scale free networks is this difference in α . Such characteristic of scale free networks observed in fig.2 could not appear unless the scaling exponent does not change.

Thus we recognize that the average of local clustering coefficient $C^{(3)}$ and transitivity or generalized clustering coefficient $C_{(3)}$ are different meanings in networks at least. It is an example that the networks considered in this subsection can uncover these differences. We further pursuit two networks introduced in this section from the perspective of disorder in the sequent subsection.

3.4 Disorder

We estimate the standard deviation and the entropy of degree distribution of two types of networks introduced in previous subsections.

Model I

Fig.8 shows that the entropy and the standard deviation decrease as the size N_0 of WS lattice becomes larger. Then the clustering coefficients $C_{(3)}$ and $C^{(3)}$ enlarged together as shown in fig.6 (A) and (D). Thus the entropy and the standard deviation decrease together when the clustering coefficients become larger. Then L increases and M_6/N in Milgram condition decreases. Overall properties in the Model I are like those of small world networks as also mentioned in the subsection 3.2.

Model II

As shown in fig.9, the standard deviation and the entropy increase as P becomes larger, when

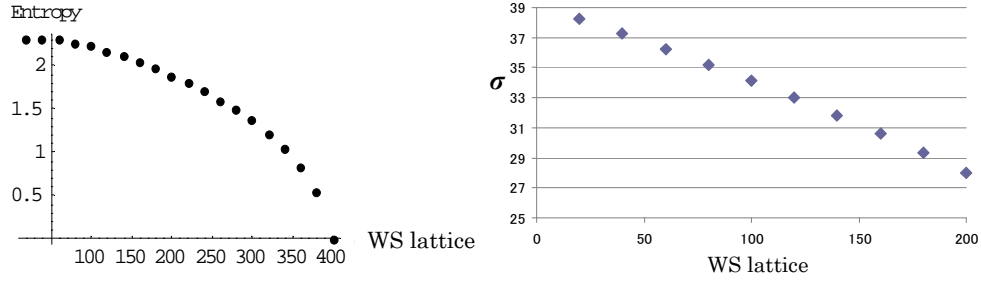


Figure 7: Entropy and standard deviation in Model (I).

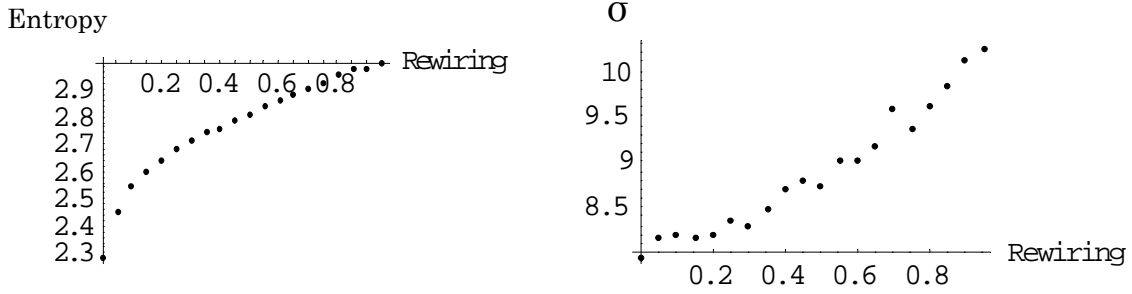


Figure 8: Entropy and Standard deviation in Model (II).

$C_{(q)}$ becomes on the whole larger in fig.7 (D). Then L decreases and M_6/N in Milgram condition increases. These properties are like those of scale free networks. It is, however, $C^{(3)}$ had a strange behavior as compared with other indices.

So far we have analyzed four types of networks, scale free networks, small world networks, Model I and Model II. The standard deviation and the entropy that are considered to measure a sort of disorder increase with M_6 in all of them. This fact implies that disorder is a crucial index for information propagation in networks.

Notice the construction of Model II only corresponds to the third step in the model introduced by B.Wang et al.[29] as mentioned above. Due to it, the variation of $C^{(3)}$ or $C_{(q)}$ is comparatively small. By exploring more various clustering tunable scale free networks such as full version of Wang et al.[29] and others proposed by [20],[21], we should scrupulously examine what is an essential index for the transition.

4 SUMMARY

Milgram condition plays an important role in the analysis of six degrees of separation[11]. We have earlier pointed out that there is a huge difference in the relation between Milgram condition and the generalized clustering coefficients in scale free networks and small world networks [9],[10]. So we found that cycle structures immanent in a network do not always play effective role for information propagation.

In this article, we first investigated the difference and further pursued what is a crucial mathematical quantity. As a result of investigating both networks, we find that a sort of "disorder", which is measured by the standard deviation of degree data and the entropy of degree distribution, plays more important role for information propagation rather than the generalized clustering coefficient $C_{(q)}$ and the average of local clustering coefficients $C_{(3)}$.

In this article, we inquired into it in detail based on further two types of intermediate networks. One is the network that link a regular network and the scale free networks introduced by Barabashi and Albert[12]. Second is the network that link a sort of small world networks and the scale free network introduced by Barabashi and Albert[12]. It would be expected that the transition between

scale free network like property and small world network like one may be clearly uncovered by the study of the proposed networks. But we could not explicitly understand it. There is a positive or negative correlation between $C_{(q)}$ and M_6 , that is, information propagation on the whole, but $C^{(3)}$ has displayed a little strange behavior as compared with the behaviors of $C_{(q)}$, L , M_6 and $C_{(q)}$. So $C^{(3)}$ may grasp a part of the transition between scale free \leftrightarrow small world (SF-SW transition). $C^{(3)}$ and $C_{(q)}$ have some different functions and important meanings, respectively, through studying such types of networks. The indices $C^{(3)}$ and $C_{(q)}$ would complement each other in our discussions apparently.

In scale free networks, it was also pointed out that the scaling exponential is crucial for information propagation. Furthermore, we found that more essential concept for information propagation is "disorder" through studying various type networks as a consequence. Then we found that small entropy of degree distribution and small standard deviation of degree data impede the propagation of information all in small world networks, scale free networks and the two types of networks proposed in this article.

So the behaviors of M_6/N or six degrees of separation in all networks discussed in this article are understood in a unified way by disorder. As a result, it turned out that the more essential quantity for information propagation is not the number of closed cycles included in a network in general but the entropy of degree distribution or standard deviation of degree data in the network. We should investigate more wide class of networks to verify our discussions.

A Basis of Power Low Distribution

We describe some basic relations which are used in this article in networks with the power low distribution of degree in this appendix.

A.1 Normalization

For scale free networks, the degree distribution with the normalization constant c is given by $P(k) = ck^{-\alpha}$. The normalization factor is determined by the following condition;

$$\sum_{k=k_{min}}^{k_{max}} cP(k) = \sum_{k=k_{min}}^{k_{max}} ck^{-\alpha} = 1, \quad (14)$$

where the minimal degree $k_{min} = 0$ and the maximal degree $k_{max} = \infty$ are taken in an ideal case but this distribution diverges at $k = 0$. Thus there must be a lower bound for k_{min} . From the lower bound k_{min} and eq. (14), we obtain

$$P(k) = \frac{k^{-\alpha}}{\zeta(\alpha, k_{min})}, \quad (15)$$

where

$$\zeta(\alpha, k_{min}) \equiv \sum_{n=0}^{\infty} (n + k_{min})^{-\alpha} \quad (16)$$

is the Hurwitz zeta function that is a generalization of the original Riemann zeta function[32]. It is, however, difficult to obtain a concrete and general expression for the Hurwitz zeta function.

The calculation, however, becomes rather simpler when a continuous approximation is made. Then the normalization constant and the entropy of degree distribution can be analytically calculated for scale free networks. First of all, we determine normalization factor of the degree distribution;

$$\int_{k=k_{min}}^{\infty} P(k)dk = \int_{k=k_{min}}^{\infty} ck^{-\alpha}dk = 1. \quad (17)$$

From eq.(17), we obtain

$$P(k) = (\alpha - 1)k_{min}^{\alpha-1} k^{-\alpha}. \quad (18)$$

Then the average degree is given by

$$\langle k \rangle = \int_{k=k_{min}}^{\infty} kP(k)dk = \frac{(\alpha - 1)k_{min}^{\alpha-1}}{2 - \alpha} [k^{2-\alpha}]_{k=k_{min}}^{\infty}. \quad (19)$$

For $\alpha > 2$, we obtain

$$\langle k \rangle = \frac{(\alpha - 1)k_{min}}{\alpha - 2}. \quad (20)$$

As to $\langle k^2 \rangle$, we obtain

$$\langle k^2 \rangle = \int_{k=k_{min}}^{\infty} k^2 P(k) dk = \frac{(\alpha - 1)k_{min}^{\alpha-1}}{3 - \alpha} [k^{3-\alpha}]_{k=k_{min}}^{\infty}. \quad (21)$$

For $\alpha > 3$, we obtain

$$\langle k^2 \rangle = \frac{(\alpha - 1)k_{min}^2}{\alpha - 3}. \quad (22)$$

Thus the valiance is obtained by

$$\langle \sigma^2 \rangle = \frac{(\alpha - 1)k_{min}^2}{(\alpha - 3)(\alpha - 2)^2}, \quad (23)$$

which has a meaning only for $\alpha > 3$.

A.2 Cut off Theory

We observed that some moments of degree k are divergent at small α in previous subsection. Since the network size, however, is finite in real networks, the maximum of degree would not be ∞ but be limited to be some value. Then the normalization factor c should be reevaluated. In continuous limit, this condition is

$$\int_{k_{min}}^{k_{max}} c k^{-\alpha} dk = 1. \quad (24)$$

As a result, we obtain

$$c = \frac{\alpha - 1}{k_{min}^{1-\alpha} - k_{max}^{1-\alpha}}. \quad (25)$$

We estimate the value k_{max} . A basic idea to do so is the relation;

$$\sum_{k_{max}}^{\infty} p(k)N \sim 1 \quad (26)$$

which means that the number of the nodes that have degree no less than k_{max} is one, roughly. In continuous limit, this condition leads to

$$\sum_{k_{max}}^{\infty} p(k)N \sim \int_{k_{max}}^{\infty} p(k)N dk = N \frac{k_{max}^{1-\alpha}}{k_{min}^{1-\alpha} - k_{max}^{1-\alpha}} \sim 1. \quad (27)$$

By noticing $k_{max} \gg k_{min}$, we obtain the relation

$$k_{max} \sim k_{min} N^{\frac{1}{\alpha-1}}. \quad (28)$$

Using this relation, the normalization factor c is obtained by

$$c = \frac{\alpha - 1}{k_{min}^{1-\alpha} - k_{max}^{1-\alpha}} \sim \frac{\alpha - 1}{k_{min}^{1-\alpha}(1 - 1/N)} \sim \frac{\alpha - 1}{k_{min}^{1-\alpha}}, \quad \text{for large } N. \quad (29)$$

Using k_{max} , we reevaluate $\langle k \rangle$ and $\langle k^2 \rangle$;

$$\langle k \rangle = \int_{k_{min}}^{k_{max}} p(k)k dk = c \frac{k_{max}^{2-\alpha} - k_{min}^{2-\alpha}}{2-\alpha}, \quad (30)$$

$$\langle k^2 \rangle = \int_{k_{min}}^{k_{max}} p(k)k^2 dk = c \frac{k_{max}^{3-\alpha} - k_{min}^{3-\alpha}}{3-\alpha}. \quad (31)$$

Substituting eq.(28) and (29) into eq. (30) and (31), we obtain

$$\langle k \rangle \sim \frac{(\alpha - 1)}{2 - \alpha} k_{min} (N^{\frac{2-\alpha}{\alpha-1}} - 1), \quad (32)$$

$$\langle k^2 \rangle \sim \frac{\alpha - 1}{(3 - \alpha)} k_{min}^2 (N^{\frac{3-\alpha}{\alpha-1}} - 1). \quad (33)$$

We now consider the cases with $\alpha > 1$, and notice

$$\frac{2-\alpha}{\alpha-1} > 1 \text{ for } 1 < \alpha < 2, \quad (34)$$

$$\frac{3-\alpha}{\alpha-1} > 1 \text{ for } 1 < \alpha < 3. \quad (35)$$

$$(36)$$

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